Kelly Criterion revisited: optimal bets

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Abstract

Kelly criterion, that maximizes the expectation value of the logarithm of wealth for bookmaker bets, gives an advantage over different class of strategies. We use projective symmetries for a explanation of this fact. Kelly's approach allows for an interesting financial interpretation of the Boltzmann/Shannon entropy. A "no-go" hypothesis for big investors is suggested.

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I. INTRODUCTION

When John L. Kelly was working for Bell Labs, he observed analogies between calculation of the optimal player's stake who enters into a gambling game and the effective transmission of information in a noisy communication channel. During the last half century the strategy which was proposed by Kelly became very popular among gamblers and inspired many authors of articles and books. The original paper dated from 1956 is hardly available. Therefore, with the AT&T consent, it has been recently reproduced in LaTeX. Today, strategies based on Kelly criterion are successfully adopted in financial markets, blackjack and even horse races. The central problem for gamblers is to find positive expectation bets. But the gambler also needs to know how to manage his money, i.e. how much to bet. Application of the Kelly criterion in blackjack was quite successful [7]. If all blackjack bets paid even money, had positive expectation and were independent, the resulting Kelly betting recipe when playing one hand at a time would be extremely simple: bet a fraction of your current capital equal to your expectation. Does the Kelly criterion unambiguously specify the winning strategy? In the thermodynamic limit the maximization of the expectation value of logarithm of the profit rate still leaves freedom of adopting different strategies. Because of calculational difficulties, only the limit case of extreme profit can be given in a concise analytical form. Kelly's association suggests a method of describing effectiveness of agents investing in the financial market in thermodynamical terms.

II. THE RULES OF THE GAME

Let us consider the simplest bookmakers bet. It can be described by disjoint alternative of two events (the random events, the majority branches of events), which we denote 1 and 2. We assume that in_k^m (where $k = 1, 2$, and $m \in \mathbb{N}$) is the fraction of current capital of m-th gambler, bet on event k, and

$$IN_k := \sum_m in_k^m > 0 \tag{1}$$

describes the sum of wagers from all the gamblers of the bet. Accordingly, out_k^m is the odds paid for the m-th gambler on the occurrence of the k-th event.

The following conditions define our bet:

A) We shall consider the case of "fair" odds (payoff odds are calculated by sharing the pool among all placed bets – the parimutuel betting), i.e.

$$\forall_{k,m} \;\; out_k^m = \alpha_k \, in_k^m \,, \tag{2}$$

where $\alpha_k \in \mathbb{R}_+$.

B) All fees and taxes are not taken into account which means that all the money is paid out to the winners:

$$IN_1 + IN_2 = \sum_m out_1^m = \sum_m out_2^m \tag{3}$$

(it should be noted that the gamblers are placing their bets differently, it means that the winners take all the pool).

The trade balance (3) is the natural premise. Let us observe that all costs and bookmaker's benefit might be the fee for participation in the game. The winner carries out an analysis of this cost after the winnings. The above condition A is equivalent to the statement that the bookmaker bet is a good offer on the effective market without an opportunity of arbitrage between the gamblers.

The conditions A and B describe uniquely the value of the factors α_k, which can be derived from formulas (1), (2) and (3). We have that:

$$\forall_k \;\; \alpha_k = \frac{IN_1 + IN_2}{IN_k} \,.$$

The formal description of the bookmaker bets with majority of branches of events might be created hierarchically as the binary tree with the leafs – elementary events, e.g. by analogy to the construction of tree-shaped key to compressing/decompressing Huffman code [2]. It follows that our binary bet is *universal*, i.e. many kinds of financial decisions we can describe as the systems based on a hierarchy of formal binary bets. Within the analogical model for insurance, the differences would only appear in the equation (4) of balanced benefit. In this case the balance (4) includes the possible loses which are relevant in insurances.

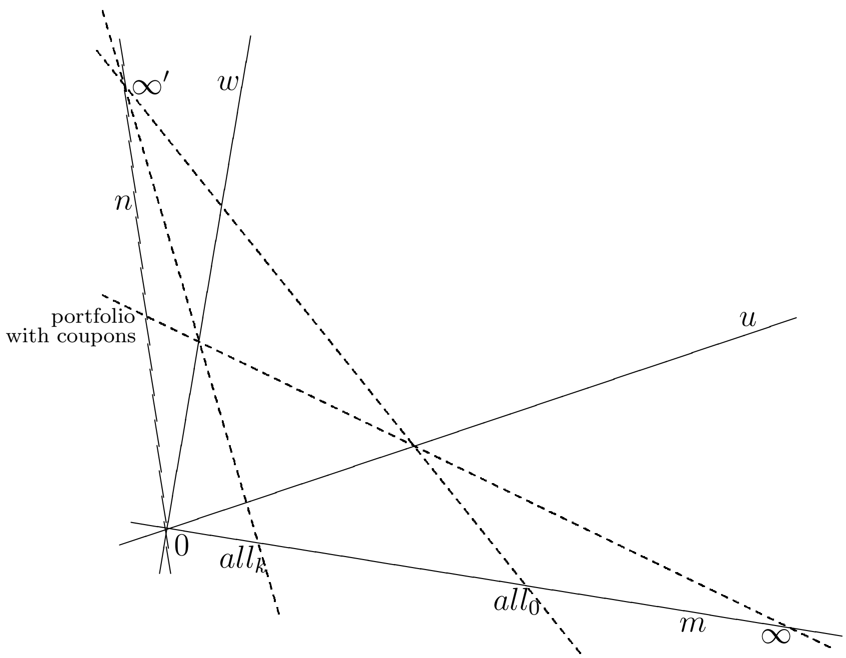
III. THE AVERAGE GAMBLER'S GAIN

We will omit the subscript m because we analyze the gain of a particular gambler. We will use the following notation: all_0 – the gambler's capital before placing bets, accordingly

all_1 – the gambler's capital after result of 1, analogically all_2 – after result of 2. The balance of expense and gambler's income is given by the formula:

$$all_k = all_0 - in_1 - in_2 + out_k \,. \tag{4}$$

From the projective geometry point of view, where the assets exchange are described without scale effect in a natural way, the profit (up to a multiplicative constant) is the unique additive invariant of the group of homographies, which include all objective irrelevant transformation between different ways of mathematical modelling of the financial effect. When the k-th event occurs the bookmaker bet, in this context, is represented by the following configuration of the straight lines [8]:



The straight lines u and w [9] express the proportion of an exchange (the market rate) of the initial capital on the our financial obligations and analogically the final obligation, respectively. The lines m and n denote the portfolio with ready money (before the closing a business and after the settlement of accounts of the bets) and the portfolio which include the bookmaker coupons (when the bet has been in effect). The set of the projection points $\{m, n\}$ is the unique invariant of the game which is defined by the gambler's strategy. The unique representation of the exchange of the bookmaker stakes u and w is possible only with the accuracy of the homograpic transformation. Thus bookmaker stakes are the covariant components of the model. They depend on choice of the basis of goods units (that means

the basis of vector space which is related to parametric portfolios – the projection points of homogeneous coordinates). Thus the set $\{m, n\}$, often called as an absolute, allows one to equip the projective space with the Hilbert metrics [1] and non-arbitrary measure of the distance between two portfolios u and w given by this metrics. It represents the profit flow in the transaction cycle $m \stackrel{u}{\to} n \stackrel{w}{\to} m$. This profit is equal to [4, 5]:

$$z_k := \ln |[n, u, w, m]| = \ln all_k - \ln all_0 \,,$$

where $[n, w, u, m]$ is a cross ratio of the projective points n, u, w, and m. Let us denote the percentage share of gambler's capital in both cases of the bookie bets by $l_k := \frac{in_k}{all_0}$ and let p_k be the probability of the k'th event. If $|in_k| \leq IN_k$ then the gambler's expected profit is equal to:

$$E(z_k)(l_1, l_2) := p_1 z_1 + p_2 z_2 = p_1 \ln(1 + \tfrac{IN_2}{IN_1} l_1 - l_2) + p_2 \ln(1 + \tfrac{IN_1}{IN_2} l_2 - l_1) \,. \qquad (5)$$

IV. MAXIMAL EXPECTED GROWTH RATE OF WEALTH

The gambler bets the stakes $\bar{l}_1$ and $\bar{l}_2$ such that her/his expected profit is the maximal one:

$$E(z_k)(\bar{l}_1, \bar{l}_2) := \max_{l_1, l_2}\{E(z)(l_1, l_2)\} \,.$$

By using the standard method we find the extremum of the differentiable function and we obtain that the family $(\bar{l}_1, \bar{l}_2) \in \mathbb{R}^2$ of the strategies solutions of above problem is described by the following straight line equation:

$$(\bar{l}_1 - p_1)\, IN_2 = (\bar{l}_2 - p_2)\, IN_1 \,, \qquad (6)$$

and the maximal profit is given by:

$$E(z_k)(\bar{l}_1, \bar{l}_2) = -\sum_{k=1,2} p_k \ln \tfrac{IN_k}{IN_2 + IN_2} - S \,, \qquad (7)$$

where $S = -\sum_k p_k \ln p_k$ is Boltzmann/Shannon entropy. Thanks to this Eq. (7), we have the financial interpretation of Kelly's formula. The maximal profit given by Eq. (7) has two components. The first of these elements is the profit on unpopularity of the winning bet (the seer's profit) $-\sum_{k=1,2} p_k \ln \frac{IN_k}{IN_2 + IN_2}$, and second means the (minus) entropy $-S$ of the branching. The value of $E(z)(\bar{l}_1, \bar{l}_2)$ is nonnegative – a rational gambler cannot loose.

Thus her/his average profit equals to 0 if the resultant preferences adopt to the probability measurement to the branching: $p_1 IN_2 = p_2 IN_1$. Consequently, one can make profit in the bookie bet only when somebody bets irrationally in the same game.

V. THE OPTIMAL STRATEGY

Till this moment we have assumed that there is no any additional condition for the simplest bookmaker bet, we allow the short position of the gamblers (negative value of $\bar{l}_k$). This is the reason why the rational gambler has the freedom of choosing the value of financial outlays $\bar{l}_1 + \bar{l}_2$ which is placed in bookmaker bets. In the absence of short positions (a typical restriction on the bet $l_1, l_2 \geq 0$) we assume that the rational gambler diversificates the risk in such a way that she/he bets only the minimal part of their resources. From all the strategies (6) we choose the optimal one:

$$(l_1^* = p_1 - \tfrac{IN_1}{IN_2}\, p_2,\ l_2^* = 0)\,,$$

when $p_1 IN_2 > p_2 IN_1$, or, equivalently, the one that can be obtained by the transposition $1 \leftrightarrow 2$ of the indices k.

If we do not have the information about proportion $\frac{IN_1}{IN_2}$ then we use Laplace's Principle of Indifference ($IN_1 = IN_2$), and in this case (when $p_1 > p_2$) the optimal stakes are (($l_1^* = p_1 - p_2,\ l_2^* = 0$), see [Kelly].

VI. BIG GAMBLERS – "NO-GO" HYPOTHESIS

Let us now consider the variant of the binary bet when the gambler's contribution of the in_k to the sum IN_k is not neglected. If the gambler pays to the pool, the pool of the bets grows from $IN_1 + IN_2$ to $(1+\delta)(IN_1 + IN_2)$, where $\delta \in \mathbb{R}$. Consequently the parts of the pool corresponding to different events are going to change from IN_k to $IN_k + \delta \frac{l_k}{l_1 + l_2}(IN_1 + IN_2)$.Then the part of the gambler's expected profit $E_\delta(z)(l_1, l_2)$ which is linear in δ will be given by:

$$E_\delta(z)(l_1,l_2) = p_1 \ln\Big(1 + \frac{\frac{IN_2}{IN_1+IN_2} + \delta \frac{l_2}{l_1+l_2}}{\frac{IN_1}{IN_1+IN_2} + \delta \frac{l_1}{l_1+l_2}}\, l_1 - l_2\Big) + p_2 \ln\Big(1 + \frac{\frac{IN_1}{IN_1+IN_2} + \delta \frac{l_1}{l_1+l_2}}{\frac{IN_2}{IN_1+IN_2} + \delta \frac{l_2}{l_1+l_2}}\, l_2 - l_1\Big) =$$

$$E(z)(l_1,l_2) + \left.\frac{\partial E_\delta(z)(l_1,l_2)}{\partial \delta}\right|_{\delta=0} \delta + O[\delta]^2\,,$$

where

$$\left.\frac{\partial E_\delta(z)(l_1,l_2)}{\partial\delta}\right|_{\delta=0} = \frac{\frac{l_1}{l_1+l_2}\,\frac{IN_1+IN_2}{IN_1}}{\frac{IN_1}{l_2\,IN_1-l_1\,IN_2}-1}\,p_1 + \frac{\frac{l_2}{l_1+l_2}\,\frac{IN_1+IN_2}{IN_2}}{\frac{IN_2}{l_1\,IN_2-l_2\,IN_1}-1}\,p_2\,. \tag{8}$$

It is sufficient to restrict oneself to the case when δ is an infinitely small number and then we can consider the corrected parameters IN_1 and IN_2 (change of δ). The extremal strategy is defined by the set of equations:

$$\frac{\partial\Big(E(z)(l_1,l_2) + \frac{\partial E_\delta(z)(l_1,l_2)}{\partial\delta}\big|_{\delta=0}\delta\Big)}{\partial l_k} = 0\,. \tag{9}$$

The solutions of these equations are the roots of two polynomials of degree five [10]. According to the fundamental theorem of Galois theory, we can conclude that an analytic form of the conditions for the optimal big player's strategy is not countable. We can find the family of parameters $(\bar{l}_1, \bar{l}_2)$ by using numerical methods, but we never can investigate the behavior of characteristics of the rational big gamblers.

Due to the universality of the binary bet model, we can conclude that any type of analysis of big investors strategies, whose appearance will disturb the financial market, will not be satisfactory because of principal mathematical reasons. It is also possible that macroeconomic thermodynamics considered as the analysis of the market disturbing strategies is forbidden by mathematics! In these contexts the tendency of diversification in the investment might be perceived as an escape of the investors from the unsolvable problems.

Is it really true that *Small Is Beautiful* [11] also in the markets?

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Appendix

The exemple code given below is written in *Mathematica 5.2* language and it allow one to calculate the algebraic expressions as a nonfactorizable polynomials of degree 5 in variables l_1 i l_2. Some zeros of these polynomials characterize optimal strategies of betting the stakes in our model of the bookmaker bet. The second polynomial can be obtained by

In[1]:= $$\text{Collect}\Big[\text{Numerator}\Big[\text{Together}\Big[\text{D}\Big[\Big(\text{Log}\Big[1-l_2+\frac{l_1 N_2}{N_1}\Big]\,p_1+\text{Log}\Big[1-l_1+\frac{l_2 N_1}{N_2}\Big]\,p_2\Big)+\Big(\frac{l_1\,(N_1+N_2)\,(l_2 N_1-l_1 N_2)\,p_1}{(l_1+l_2)\,N_1\,(-(-1+l_2)\,N_1+l_1 N_2)}+\frac{l_2\,(N_1+N_2)\,(l_2 N_1-l_1 N_2)\,p_2}{(l_1+l_2)\,N_2\,(-l_2 N_1+(-1+l_1)\,N_2)}\Big)\,\delta,\; l_1\Big]\Big]\Big],\,\{l_1\},\,\text{Factor}\Big]\;/.\;\{p_1+p_2\to 1\}$$

Out[1]= $$\begin{aligned}
&l_1^5 N_1 N_2^5 - \\
&l_1^4 N_2^4\,(-N_1^2 p_1+\delta N_1^2 p_1+3\,l_2 N_1^2 p_1+2\,N_1 N_2 p_1+\delta N_1 N_2 p_1-2\,l_2 N_1 N_2 p_1+\delta\,l_2 N_1 N_2 p_1+\delta\,l_2 N_2^2 p_1- \\
&\quad 2\,N_1^2 p_2+3\,l_2 N_1^2 p_2-\delta\,l_2 N_1^2 p_2+N_1 N_2 p_2-2\,l_2 N_1 N_2 p_2-\delta\,l_2 N_1 N_2 p_2)+ \\
&l_1^3 N_2^3\,(-2\,l_2 N_1^3 p_1+2\,\delta\,l_2 N_1^3 p_1+3\,l_2^2 N_1^3 p_1-2\,N_1^2 N_2 p_1+2\,\delta N_1^2 N_2 p_1+6\,l_2 N_1^2 N_2 p_1- \\
&\quad 6\,l_2^2 N_1^2 N_2 p_1+4\,\delta\,l_2^2 N_1^2 N_2 p_1+N_1 N_2^2 p_1+2\,\delta N_1 N_2^2 p_1-4\,l_2 N_1 N_2^2 p_1+l_2^2 N_1 N_2^2 p_1+4\,\delta\,l_2^2 N_1 N_2^2 p_1+ \\
&\quad 2\,\delta\,l_2 N_2^3 p_1+N_1^3 p_2-4\,l_2 N_1^3 p_2+2\,\delta\,l_2 N_1^3 p_2+3\,l_2^2 N_1^3 p_2-4\,\delta\,l_2^2 N_1^3 p_2-2\,N_1^2 N_2 p_2+6\,l_2 N_1^2 N_2 p_2+ \\
&\quad 2\,\delta\,l_2 N_1^2 N_2 p_2-6\,l_2^2 N_1^2 N_2 p_2-4\,\delta\,l_2^2 N_1^2 N_2 p_2-2\,l_2 N_1 N_2^2 p_2+l_2^2 N_1 N_2^2 p_2)+(-1+l_2)\,l_2^2 \\
&N_1^2\,(-\delta\,l_2^2 N_1^3 N_2 p_1-2\,\delta\,l_2 N_1^2 N_2^2 p_1-l_2^2 N_1^2 N_2^2 p_1-\delta\,l_2^2 N_1^2 N_2^2 p_1-\delta N_1 N_2^3 p_1-2\,l_2 N_1 N_2^3 p_1- \\
&\quad 2\,\delta\,l_2 N_1 N_2^3 p_1-N_2^4 p_1-\delta N_2^4 p_1-\delta\,l_2 N_1^4 p_2+\delta\,l_2^2 N_1^4 p_2-\delta N_1^3 N_2 p_2+\delta\,l_2^2 N_1^3 N_2 p_2-2\,\delta N_1^2 N_2^2 p_2+ \\
&\quad l_2 N_1^2 N_2^2 p_2+2\,\delta\,l_2 N_1^2 N_2^2 p_2-l_2^2 N_1^2 N_2^2 p_2+N_1 N_2^3 p_2-\delta N_1 N_2^3 p_2-l_2 N_1 N_2^3 p_2+\delta\,l_2 N_1 N_2^3 p_2)- \\
&l_1^2 N_2^2\,(-l_2^2 N_1^4 p_1+\delta\,l_2^2 N_1^4 p_1+l_2^3 N_1^4 p_1-2\,l_2 N_1^3 N_2 p_1+2\,\delta\,l_2 N_1^3 N_2 p_1+6\,l_2^2 N_1^3 N_2 p_1- \\
&\quad 4\,\delta\,l_2^2 N_1^3 N_2 p_1-6\,l_2^3 N_1^3 N_2 p_1+6\,\delta\,l_2^3 N_1^3 N_2 p_1-N_1^2 N_2^2 p_1+\delta N_1^2 N_2^2 p_1+5\,l_2 N_1^2 N_2^2 p_1- \\
&\quad 2\,\delta\,l_2 N_1^2 N_2^2 p_1-9\,l_2^2 N_1^2 N_2^2 p_1+\delta\,l_2^2 N_1^2 N_2^2 p_1+3\,l_2^3 N_1^2 N_2^2 p_1+6\,\delta\,l_2^3 N_1^2 N_2^2 p_1+\delta N_1 N_2^3 p_1- \\
&\quad 2\,l_2 N_1 N_2^3 p_1-3\,\delta\,l_2 N_1 N_2^3 p_1+2\,l_2^2 N_1 N_2^3 p_1+6\,\delta\,l_2^2 N_1 N_2^3 p_1+\delta\,l_2 N_2^4 p_1+l_2 N_1^4 p_2- \\
&\quad \delta\,l_2 N_1^4 p_2-2\,l_2^2 N_1^4 p_2+6\,\delta\,l_2^2 N_1^4 p_2+l_2^3 N_1^4 p_2-6\,\delta\,l_2^3 N_1^4 p_2+N_1^3 N_2 p_2-4\,l_2 N_1^3 N_2 p_2- \\
&\quad \delta\,l_2 N_1^3 N_2 p_2+9\,l_2^2 N_1^3 N_2 p_2+5\,\delta\,l_2^2 N_1^3 N_2 p_2-6\,l_2^3 N_1^3 N_2 p_2-6\,\delta\,l_2^3 N_1^3 N_2 p_2+ \\
&\quad 4\,l_2 N_1^2 N_2^2 p_2-6\,l_2^2 N_1^2 N_2^2 p_2-2\,\delta\,l_2^2 N_1^2 N_2^2 p_2+3\,l_2^3 N_1^2 N_2^2 p_2+l_2^2 N_1 N_2^3 p_2-\delta\,l_2^2 N_1 N_2^3 p_2)- \\
&l_1 l_2 N_1 N_2\,(-2\,l_2^2 N_1^3 N_2 p_1+4\,\delta\,l_2^2 N_1^3 N_2 p_1+2\,l_2^3 N_1^3 N_2 p_1-4\,\delta\,l_2^3 N_1^3 N_2 p_1-4\,l_2 N_1^2 N_2^2 p_1+ \\
&\quad 6\,\delta\,l_2 N_1^2 N_2^2 p_1+6\,l_2^2 N_1^2 N_2^2 p_1-2\,\delta\,l_2^2 N_1^2 N_2^2 p_1-3\,l_2^3 N_1^2 N_2^2 p_1-4\,\delta\,l_2^3 N_1^2 N_2^2 p_1- \\
&\quad 2\,N_1 N_2^3 p_1+2\,\delta N_1 N_2^3 p_1+4\,l_2 N_1 N_2^3 p_1+4\,\delta\,l_2 N_1 N_2^3 p_1-4\,l_2^2 N_1 N_2^3 p_1-6\,\delta\,l_2^2 N_1 N_2^3 p_1+ \\
&\quad 2\,\delta N_2^4 p_1-l_2 N_2^4 p_1-2\,\delta\,l_2 N_2^4 p_1+2\,\delta\,l_2 N_1^4 p_2-6\,\delta\,l_2^2 N_1^4 p_2+4\,\delta\,l_2^3 N_1^4 p_2+ \\
&\quad 2\,l_2 N_1^3 N_2 p_2-4\,l_2^2 N_1^3 N_2 p_2-4\,\delta\,l_2^2 N_1^3 N_2 p_2+2\,l_2^3 N_1^3 N_2 p_2+4\,\delta\,l_2^3 N_1^3 N_2 p_2+ \\
&\quad 2\,N_1^2 N_2^2 p_2-5\,l_2 N_1^2 N_2^2 p_2-4\,\delta\,l_2 N_1^2 N_2^2 p_2+6\,l_2^2 N_1^2 N_2^2 p_2+4\,\delta\,l_2^2 N_1^2 N_2^2 p_2- \\
&\quad 3\,l_2^3 N_1^2 N_2^2 p_2+2\,l_2 N_1 N_2^3 p_2-2\,\delta\,l_2 N_1 N_2^3 p_2-2\,l_2^2 N_1 N_2^3 p_2+2\,\delta\,l_2^2 N_1 N_2^3 p_2)
\end{aligned}$$

the transposition $1 \leftrightarrow 2$ of the indices k.

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[6] W. Poundstone, *Fortune's Formula: The Untold Story of the Scientific Betting System That Beat the Casinos and Wall Street*, Hill & Wang, New York, 2005.

[7] E. O. Thorp, *The Kelly criterion in blackjack, sports betting, and the stock market*, paper presented at The 10th International Conference on Gambling and Risk Taking, June 1997, revised May 1998, in http://www.bjmath.com/bjmath/

[8] Or the dual configuration.

[9] On the market of goods, the lines w and u represent the hyperplane of codimension one.

[10] We do not give their explicit form, because they can be easily generated by using the set of equations (5), (8), and (9), and taking advantage of the language symbolical calculation *Mathematica*. Few exemplary lines are added in Appendix

[11] This is a title of the cult book of Fritz Schumacher.